Differential Geometry III: Riemannian geometry

5.1 (a) Let $X \in \Gamma(\mathcal{M})$ be a *Killing vector field* on the Riemannian manifold $(\mathcal{M}, g)$. Show that, for any $V, W \in \Gamma(\mathcal{M})$,

$$g\big(\nabla_V X, W\big) + g\big(\nabla_W X, V\big) = 0,$$

where ∇ is the Levi-Civita connection of g *(Hint: Compute the derivative $X(g(V,W))$ in two ways, once viewing it as a Lie derivative and another using the connection)*. Deduce that, if $\gamma : (a,b) \to \mathcal{M}$ is a geodesic of $(\mathcal{M}, g)$, then the function $t \to g(X|_{\gamma(t)}, \dot{\gamma})$ is constant for $t \in (a,b)$

Remark. Any function $F : T\mathcal{M} \to \mathbb{R}$ such that $F(\gamma(t), \dot{\gamma}(t))$ is constant when γ is a geodesic is called a *constant of motion* for the geodesic flow.

(b) Let $\zeta : (-1,1) \to \mathbb{R}^2$, $\zeta(u) = \big(x(u), y(u)\big)$ be a smooth curve parametrized with *unit speed* and contained in the upper half plane, i.e. $y(u) > 0$ for all $u \in (-1,1)$. Let S be the *surface of revolution* in $\mathbb{R}^3$ obtained by rotating the curve ζ around the x-axis, i.e. S is parametrized by the map $\Psi : (-1,1) \times [0, 2\pi)$,

$$\Psi(u, \varphi) = \big(x(u), y(u)\cos(\varphi), y(u)\sin(\varphi)\big).$$

Let also g be the metric induced on S by the Euclidean metric g_E in $\mathbb{R}^3$. Show that the vector field $\Phi = \frac{\partial}{\partial \varphi}$ is a Killing vector field on (S, g). Find a closed formula for any geodesic $\gamma : (-T, T) \to (S, g)$, $t \to \big(u(t), \varphi(t)\big)$ *(Hint: Use the fact that $g(\dot{\gamma}, \Phi)$ and $g(\dot{\gamma}, \dot{\gamma})$ are conserved along γ to obtain a simple expression for $\dot{\gamma} = (\frac{du}{dt}, \frac{d\varphi}{dt})$).*

5.2 The *Poincaré half-plane* is the domain $\mathbb{H}^2 = \big\{(x,y) \in \mathbb{R}^2 : y > 0\big\}$ in $\mathbb{R}^2$ equipped with the Riemannian metric

$$g_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}.$$

(a) Setting $z = x + iy$, show that the map $f : \mathbb{H}^2 \to \mathbb{H}^2$ defined by

$$f(z) = \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$ is an *isometry* for $g_{\mathbb{H}}$.

(b) Show that the geodesic equation for $g_{\mathbb{H}}$ takes the form

$$\ddot{x} = \frac{2\dot{x}\dot{y}}{y}, \quad \ddot{y} = \frac{\dot{y}^2 - \dot{x}^2}{y}.$$

(c) Show that $\frac{\dot{x}^2 + \dot{y}^2}{y^2}$ and $\frac{\dot{x}}{y^2}$ are constant along a geodesic (i.e. are constants of motion for the geodesic flow). Is the conserved quantity $\frac{\dot{x}}{y^2}$ a constant of motion associated to a Killing vector field of $(\mathbb{H}^2, g_{\mathbb{H}})$, in the spirit of Exercise 5.1? What is the shape of a geodesic curve in $(\mathbb{H}^2, g_{\mathbb{H}})$?

Remark. The *Poincaré half-plane* is a model for the *hyperbolic plane*.

5.3 Let $(\mathcal{M}^n, g)$ be a smooth Riemannian manifold and let $\gamma : [0,1] \to \mathcal{M}$ be a geodesic.

(a) Show that there exist a set of vector fields $\{E_i\}_{i=1}^n$ defined along the curve γ satisfying all of the following conditions:

- For any $t \in [0,1]$, the tangent vectors $\{E_i|_{\gamma(t)}\}_{i=1}^n$ at $\gamma(t)$ form an orthonormal basis of $T_{\gamma(t)}\mathcal{M}$.
- $E_1|_{\gamma(t)}$ is parallel to $\dot\gamma(t)$.
- The vector fields E_i are parallel translated along γ, i.e. $\nabla_{\dot\gamma} E_i = 0$, $i = 1, \dots, n$.

(b) Show that, if $\{E_i\}_{i=1}^n$ is a set of vector fields along γ as above and X is any other vector field along γ, then X is parallel-translated along γ if and only if the components of $X|_{\gamma(t)}$ in the basis $\{E_i|_{\gamma(t)}\}_{i=1}^n$ of $T_{\gamma(t)}\mathcal{M}$ are constant as functions of t.

5.4 Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $\Psi : \Omega \to \mathbb{R}^N$ $(N > n)$ be a smooth immersion; let also g be the Riemannian metric induced on Ω by the Euclidean metric g_E on $\mathbb{R}^N$. Recall that, for for any point $p \in \Omega$ and any local coordinate system $(x^1, \dots, x^n)$ around p, the tangent space $T_{\Psi(p)}\Psi(\Omega)$ of the sumbanifold $\Psi(\Omega) \subset \mathbb{R}^N$ (i.e. the image of the map $d\Psi_p : T_p\Omega \to T_{\Psi(p)}\mathbb{R}^N$) is spanned by the vectors $\{\partial_i \Psi\}_{i=1}^n$. Let us denote with $\Pi^\top_{\Psi(p)} : T_{\Psi(p)}\mathbb{R}^N \to T_{\Psi(p)}\Psi(\Omega)$ the *orthogonal projection* with respect to the Euclidean inner product on $T_{\Psi(p)}\mathbb{R}^N$. Show that the Christoffel symbols of the Levi-Civita connection for g in the *Cartesian* coordinate system $(x^1, \dots, x^n)$ on $\Omega \subset \mathbb{R}^n$ satisfy for any $p \in \Omega$:

$$\Pi^\top_{\Psi(p)}\Big(\frac{\partial^2 \Psi}{\partial x^i \partial x^j}(p)\Big) = \Gamma^k_{ij}(p)\partial_k \Psi(p).$$